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Unified streamline, heatline and massline methods for the visualization of two-dimensional heat and mass transfer in anisotropic media

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Abstract

Many of the actual materials are anisotropic, ranging from natural products to the most sophisticated composite materials. Special emphasis needs to be devoted to the heat and mass transfer calculations in anisotropic media, and to the development of visualization tools for the transport phenomena occurring in such media, similarly to what happens with isotropic media. The most adequate tools for visualization purposes are the streamlines, the heatlines and the masslines, when dealing with two-dimensional steady problems without source terms. Moreover, further attention needs to be devoted to the diffusion coefficients for the streamfunction, heatfunction and massfunction, whose contour plots are used for visualization purposes. This is specially important for domains of marked anisotropy or for domains involving media of different properties, or even conjugate diffusion/convection heat and mass transfer problems. Once defined the proper diffusion coefficients, it is proposed a unified physical treatment, as well as a unified treatment to evaluate the function's fields by using the same numerical procedures and code routines as for the primitive conserved variables. The unified approach is illustrated through pure conduction heat transfer problems, natural convection heat transfer in a porous enclosure, and conjugate conduction–convection heat transfer.

1. Introduction

The heatflux lines are well established to visualize heat transfer by pure conduction in isotropic media [1]. The concept evolved to the heatlines, through the introduction of the heatfunction in the 80's, its application extending to the field of convection heat transfer [2,3]. Many examples of heatline applications in convective heat transfer can be found in the literature [2–15], as well as in some conjugate conduction-convection heat transfer problems in isotropic media [16–18]. A natural extension of the method was made to the field of convective mass transfer through the introduction of the massfunction and massline concepts [19]. These have been applied to pure convective mass transfer [14,19] and conjugate diffusion-convection mass transfer in isotropic media [16–18], as well as to convection heat transfer in isotropic porous media [13,14].

From the physical and numerical viewpoints, the streamline, heatline and massline concepts have been unified to deal with isotropic media [18], with the proper diffusion coefficients for the streamfunction, heatfunction and massfunction. Such diffusion coefficients have been involved in some discussion for domains with portions of different transport properties [20,21]. An extension was made to the use of the heatlines and masslines in reacting flows [22], the conserved scalar variables being the total enthalpy (sum of sensible enthalpy and enthalpy of reaction) and the atomic mass fractions of the individual elements. The natural evolution of the work reported in [18] is the unification of the streamline, heatline and massline methods to apply to anisotropic media, where special care needs to be devoted to the diffusion coefficients for the streamfunction, heatfunction and massfunction. This is the objective of the present work.

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Nomenclature

c_p	constant pressure specific heat
C	concentration
D	mass diffusivity
g	gravitational acceleration
H	heatfunction
J	transport flux
k	thermal conductivity
K	permeability
l	direction co-sine
L	length (height)
m	direction co-sine
M	massfunction
p	pressure
R	ratio between the transfer properties
Ra	Darcy Rayleigh number
s	surface boundary
S	source term
T	temperature
u, v	Cartesian velocity components
x, y	Cartesian co-ordinates

Greek symbols

α	thermal diffusivity
β	volumetric expansion coefficient
Γ	generic diffusion coefficient
γ	anisotropy angle: thermal conductivity
Δ	difference value
δ	anisotropy angle: permeability
μ	dynamic viscosity
ν	kinematic viscosity
ρ	density
ϕ	generic intensive variable
Φ	generic function for visualization
ψ	streamfunction

Subscripts

b	at the boundary
C	cold (lower value)
H	hot (higher value)
i	chemical mass species i
k	referring thermal conductivity
K	referring permeability
P	at local boundary point P
x, y	referring Cartesian co-ordinates
0	reference value
1,2	indices of direction co-sines
1,2	portions with different properties
ξ, η	referring direction co-sines
ϕ	referring generic variable ϕ
Φ	referring generic function Φ
ψ	referring streamfunction
$*$	dimensionless

The mostly used numerical methods to solve heat and mass transfer problems are control-volume based, being thus developed for conserved variables. However, they can be also used for variables that are not conserved, but whose differential equations are formally similar to that of the conserved variables, as it is the case of the streamfunction, heatfunction and massfunction. Thus, all the variables, conserved or not, can be evaluated using the same numerical procedures and code routines as developed for the primitive conserved variables.

The involved anisotropic media can be a solid, where pure heat conduction or/and pure mass diffusion occurs, or a porous media where fluid flows (Darcy flow) and convective heat or/and mass (chemical species) transfer occurs. The global domain under analysis can be composed of solid and porous portions, as well as of anisotropic and isotropic portions. The isotropic situation is just a particular case of the general anisotropic one.

2. Physical modeling

2.1. General formulation

The usual steady two-dimensional heat and mass transfer phenomena can be expressed in terms of fluxes through the general differential equation,

$$\frac{\partial}{\partial x}(J_{\phi,x}) + \frac{\partial}{\partial y}(J_{\phi,y}) = S_\phi \tag{1}$$

where ϕ is the generic primitive conserved variable. In the present work, attention will be given to the situations with null source term, $S_\phi = 0$, the unique situation for which the functions and lines under analysis make sense. Situations with non-zero source terms for a particular ϕ cannot be treated with the visualization tools developed in the present work.

As the media under analysis are anisotropic, the fluxes in Eq. (1) are expressed as

$$J_{\phi,x} = \rho u(\phi - \phi_0) - \left[(\Gamma_{\phi,\xi} l_1^2 + \Gamma_{\phi,\eta} l_2^2)\frac{\partial \phi}{\partial x} + (\Gamma_{\phi,\xi} l_1 m_1 + \Gamma_{\phi,\eta} l_2 m_2)\frac{\partial \phi}{\partial y}\right] \tag{2}$$

$$J_{\phi,y} = \rho v(\phi - \phi_0) - \left[(\Gamma_{\phi,\xi} l_1 m_1 + \Gamma_{\phi,\eta} l_2 m_2)\frac{\partial \phi}{\partial x} + (\Gamma_{\phi,\xi} m_1^2 + \Gamma_{\phi,\eta} m_2^2)\frac{\partial \phi}{\partial y}\right] \tag{3}$$

where u and v are the area averaged Cartesian velocity components [3], $\Gamma_{\phi,\xi}$ and $\Gamma_{\phi,\eta}$ are the principal diffusion coefficients for ϕ along the ξ, η principal Cartesian directions, which are related with the x, y Cartesian co-ordinate system through the direction cosines l_1, l_2 and m_1, m_2. ϕ_0 is a reference value for ϕ, taken as its lower value in the entire domain [14,18]. With $\Gamma_{\phi,xx} = \Gamma_{\phi,\xi} l_1^2 + \Gamma_{\phi,\eta} l_2^2$, $\Gamma_{\phi,yy} = \Gamma_{\phi,\xi} m_1^2 + \Gamma_{\phi,\eta} m_2^2$ and $\Gamma_{\phi,xy} = \Gamma_{\phi,yx} = \Gamma_{\phi,\xi} l_1 m_1 + \Gamma_{\phi,\eta} l_2 m_2$, Eqs. (2) and (3) come

$$J_{\phi,x} = \rho u(\phi - \phi_0) - \left(\Gamma_{\phi,xx} \frac{\partial \phi}{\partial x} + \Gamma_{\phi,xy} \frac{\partial \phi}{\partial y} \right) \tag{4}$$

$$J_{\phi,y} = \rho v(\phi - \phi_0) - \left(\Gamma_{\phi,xy} \frac{\partial \phi}{\partial x} + \Gamma_{\phi,yy} \frac{\partial \phi}{\partial y} \right) \tag{5}$$

Inserting these fluxes in Eq. (1), with $S_\phi = 0$, one obtains the general convection–diffusion differential transport equation for ϕ in two-dimensional anisotropic media

$$\frac{\partial}{\partial x} \left[\rho u \phi - \left(\Gamma_{\phi,xx} \frac{\partial \phi}{\partial x} + \Gamma_{\phi,xy} \frac{\partial \phi}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[\rho v \phi - \left(\Gamma_{\phi,xy} \frac{\partial \phi}{\partial x} + \Gamma_{\phi,yy} \frac{\partial \phi}{\partial y} \right) \right] = 0 \tag{6}$$

If the fluid flow subsides (stagnant fluid or solid medium, $u = v = 0$), this corresponds to a pure diffusion situation. The energy equation can be considered if there are no source or sink terms, and the i species mass conservation equation can be considered if there is no i species production or destruction (no chemical reactions involving i species). Some particular meanings of ϕ are presented in Table 1, the equations being considered in their conservative form, as they are to be solved using numerical methods developed for that purpose. The global mass conservation ($\phi = 1$), with null diffusion coefficients and null source term, is valid for any medium without nuclear reactions as in the present case, becoming $\partial/\partial x(\rho u) + \partial/\partial y(\rho v) = 0$. The terms involving ϕ_0 vanish in Eq. (6), by invoking the global mass conservation equation.

When dealing with saturated porous media, the velocity components are related to the pressure gradient and the body force (in this case only the gravitational acceleration, being the x, y co-ordinate system placed such that $g_x = 0$ and $g_y = -g$) through the Darcy flow model, which is sufficient for many practical applications [23]

$$u = -\frac{1}{\mu} \left[K_{xx} \frac{\partial p}{\partial x} + K_{xy} \left(\frac{\partial p}{\partial y} + \rho g \right) \right]$$
$$v = -\frac{1}{\mu} \left[K_{xy} \frac{\partial p}{\partial x} + K_{yy} \left(\frac{\partial p}{\partial y} + \rho g \right) \right] \tag{7}$$

However, if there are interfaces between fluid saturated porous media and pure fluids (the latter governed by the complete Navier–Stokes equations), or the Reynolds number is high enough such that inertial effects must be taken into account, more detailed and complete models (usually the Brinkman and Forchheimer modifications, respectively) are needed [23].

Inserting the velocity components as given by Eq. (7) into the overall mass conservation equation, expressed as $\partial/\partial x(\rho u) + \partial/\partial y(\rho v) = 0$, a diffusion equation is obtained with the pressure as the main dependent variable. It should be noted that p is not a conserved variable, the diffusion equation for p appearing due to the use of the Darcy flow model. If the properties are constant or the gravity influence is absent, the source term of the differential equation for p vanishes. If natural convection is to be modeled using the Boussinesq approach, only the density directly related with the gravity acceleration should be taken as dependent on the temperature, ρ^2 in the last row of Table 1 being expressed as $\rho^2 = \rho_0\{\rho_0[1- \beta(T - T_0)]\}$.

The function $\Phi(x, y)$, whose contour plots are used for visualization purposes, is defined through its first order derivatives as

$$\frac{\partial \Phi}{\partial y} = J_{\phi,x} = \rho u(\phi - \phi_0) - \left[\Gamma_{\phi,xx} \frac{\partial \phi}{\partial x} + \Gamma_{\phi,xy} \frac{\partial \phi}{\partial y} \right] \tag{8}$$

$$-\frac{\partial \Phi}{\partial x} = J_{\phi,y} = \rho v(\phi - \phi_0) - \left[\Gamma_{\phi,xy} \frac{\partial \phi}{\partial x} + \Gamma_{\phi,yy} \frac{\partial \phi}{\partial y} \right] \tag{9}$$

Equating the second order cross derivatives of Φ, being implicitly assumed that it is a continuous function to its second order derivatives, Eq. (6) is identically obtained.

The total differential of Φ is obtained as

$$\mathrm{d}\Phi = \frac{\partial \Phi}{\partial x} \mathrm{d}x + \frac{\partial \Phi}{\partial y} \mathrm{d}y = -J_{\phi,y} \mathrm{d}x + J_{\phi,x} \mathrm{d}y \tag{10}$$

If $\mathrm{d}\Phi = 0$ along a given segment $\mathrm{d}s$, such that $(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$, it means that there is no ϕ flow crossing such segment, that is, a Φ constant line is not crossed by the ϕ flow, being thus a line that is tangent to

Table 1
Physical principles, diffusion coefficients and source terms for different particular meanings of ϕ

Physical principle	ϕ	$\Gamma_{\phi,xx}$	$\Gamma_{\phi,yy}$	$\Gamma_{\phi,xy}$	S_ϕ
Overall mass conservation	1	0	0	0	0
i species mass conservation	C_i	$\rho D_{i,xx}$	$\rho D_{i,yy}$	$\rho D_{i,xy}$	0
Energy conservation	T	k_{xx}/c_p	k_{yy}/c_p	k_{xy}/c_p	0
Overall mass conservation (Darcy flow model)	p	$\rho K_{xx}/\mu$	$\rho K_{yy}/\mu$	$\rho K_{xy}/\mu$	$\frac{\partial}{\partial x}\left(\frac{\rho^2 g K_{xy}}{\mu}\right) + \frac{\partial}{\partial y}\left(\frac{\rho^2 g K_{yy}}{\mu}\right)$

the flow vector [18]. A difference $\Delta\Phi$ between the Φ values at two points represents the ϕ flow, which crosses the segment linking these points. This is especially instructive because it illustrates the streets comprised between two constant Φ lines, along which are transferred the ϕ flows. The lines of constant Φ (the contour plots of function Φ) are the most effective tools for visualization purposes of the transport phenomena related with the corresponding ϕ variable [18].

From Eqs. (8) and (9) it can be obtained that

$$\frac{\partial\phi}{\partial x} = -\frac{\Gamma_{\phi,yy}}{\Gamma^2}\frac{\partial\Phi}{\partial y} - \frac{\Gamma_{\phi,xy}}{\Gamma^2}\frac{\partial\Phi}{\partial x} + \frac{\Gamma_{\phi,yy}}{\Gamma^2}\rho u(\phi-\phi_0) - \frac{\Gamma_{\phi,xy}}{\Gamma^2}\rho v(\phi-\phi_0) \quad (11)$$

$$\frac{\partial\phi}{\partial y} = \frac{\Gamma_{\phi,xx}}{\Gamma^2}\frac{\partial\Phi}{\partial x} - \frac{\Gamma_{\phi,xy}}{\Gamma^2}\frac{\partial\Phi}{\partial y} + \frac{\Gamma_{\phi,xx}}{\Gamma^2}\rho v(\phi-\phi_0) - \frac{\Gamma_{\phi,xy}}{\Gamma^2}\rho u(\phi-\phi_0) \quad (12)$$

where

$$\Gamma^2 = \Gamma_{\phi,xx}\Gamma_{\phi,yy} - \Gamma^2_{\phi,xy} \quad (13)$$

From irreversible thermodynamics, it is stated that $\Gamma_{\phi,\xi} > 0$ and $\Gamma_{\phi,\eta} > 0$ [24,25]. If the co-ordinate systems are such that the system x, y is obtained from the (principal) system ξ, η by a rotation of amplitude γ, the direction cosines are $l_1 = \cos\gamma$, $l_2 = -\sin\gamma$, $m_1 = \sin\gamma$ and $m_2 = \cos\gamma$. It comes then that $\Gamma_{\phi,xx} = \Gamma_{\phi,\xi}\cos^2\gamma + \Gamma_{\phi,\eta}\sin^2\gamma$, $\Gamma_{\phi,yy} = \Gamma_{\phi,\xi}\sin^2\gamma + \Gamma_{\phi,\eta}\cos^2\gamma$, $\Gamma_{\phi,xy} = (\Gamma_{\phi,\xi} - \Gamma_{\phi,\eta})\sin\gamma\cos\gamma$ and $\Gamma^2 = \Gamma_{\phi,\xi}\Gamma_{\phi,\eta}$, being thus, effectively, $\Gamma^2 > 0$.

Assuming now that ϕ is a continuous function to its second order derivatives, it can be established the equality of its second order cross derivatives, obtained from Eqs. (11) and (12), leading to the equation

$$\begin{aligned}0 = {} & \frac{\partial}{\partial x}\left(\frac{\Gamma_{\phi,xx}}{\Gamma^2}\frac{\partial\Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\Gamma_{\phi,yy}}{\Gamma^2}\frac{\partial\Phi}{\partial y}\right) \\ & + \left[\frac{\partial}{\partial x}\left(\frac{\Gamma_{\phi,xy}}{\Gamma^2}\frac{\partial\Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\Gamma_{\phi,xy}}{\Gamma^2}\frac{\partial\Phi}{\partial x}\right)\right] \\ & + \left\{\frac{\partial}{\partial x}\left[\frac{\rho(\phi-\phi_0)}{\Gamma^2}(\Gamma_{\phi,xx}v - \Gamma_{\phi,xy}u)\right] \right. \\ & \left. - \frac{\partial}{\partial y}\left[\frac{\rho(\phi-\phi_0)}{\Gamma^2}(\Gamma_{\phi,yy}u - \Gamma_{\phi,xy}v)\right]\right\} \end{aligned} \quad (14)$$

This is a diffusion equation for Φ in anisotropic media, with the source term (present in its second row). It is evident from Eq. (14) that the diffusion coefficients for Φ are

$$\Gamma_{\Phi,xx} = \frac{\Gamma_{\phi,xx}}{\Gamma^2} \quad \Gamma_{\Phi,yy} = \frac{\Gamma_{\phi,yy}}{\Gamma^2} \quad \Gamma_{\Phi,xy} = \frac{\Gamma_{\phi,xy}}{\Gamma^2} \quad (15)$$

If the x, y co-ordinate system is coincident with the principal system, it is $\Gamma_{\Phi,xx} = \Gamma_{\Phi,\xi} = 1/\Gamma_{\phi,\eta}$ and $\Gamma_{\Phi,yy} = \Gamma_{\Phi,\eta} = 1/\Gamma_{\phi,\xi}$, i.e., the principal diffusion coefficients for Φ are the inverse of the diffusion coefficient for ϕ in the perpendicular directions. For isotropic media, it comes that $\Gamma_\Phi = 1/\Gamma_\phi$, as proposed in [18].

Similarly to what was made to obtain the differential equation for Φ, defining the streamfunction ψ (overall massfunction) through its first order derivatives as

$$\partial\psi/\partial y = \rho u \quad -\partial\psi/\partial x = \rho v \quad (16)$$

it is obtained the diffusion equation for ψ,

$$\begin{aligned}0 = {} & \frac{\partial}{\partial x}\left(\frac{\mu K_{xx}}{\rho K^2}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\mu K_{yy}}{\rho K^2}\frac{\partial\psi}{\partial y}\right) \\ & + \left[\frac{\partial}{\partial x}\left(\frac{\mu K_{xy}}{\rho K^2}\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\mu K_{xy}}{\rho K^2}\frac{\partial\psi}{\partial x}\right)\right] - \frac{\partial}{\partial x}(\rho g) \end{aligned} \quad (17)$$

where $K^2 = K_{xx}K_{yy} - K^2_{xy}$, which is formally similar to Eq. (14). It is evident that the diffusion coefficients for ψ are

$$\Gamma_{\psi,xx} = \frac{\mu K_{xx}}{\rho K^2} \quad \Gamma_{\psi,yy} = \frac{\mu K_{yy}}{\rho K^2} \quad \Gamma_{\psi,xy} = \frac{\mu K_{xy}}{\rho K^2} \quad (18)$$

The source term in Eq. (17), $-\partial/\partial x(\rho g)$, vanishes in the case of constant property fluids or absent gravity effects, but it is a driving term in natural or mixed convection situations.

Eqs. (6), (14) and (17) are formally similar (the differential equations for Φ and for ψ having null convective terms) and, for each particular Φ, its solution can be obtained by following the same procedures as for ϕ, once the boundary conditions are established. In this way, Φ is treated as a conserved variable even if it is not a physically conserved variable. From a physical viewpoint, it is thus proposed an unified treatment for the functions whose contour plots are used for visualization purposes when dealing with transport phenomena in two-dimensional anisotropic media.

The coupling between ϕ and Φ, as well as the coupling between the diffusion coefficients for ϕ and for Φ, and the particular meaning of Φ for some usual situations are summarized in Table 2.

2.2. Boundary conditions

The function Φ is defined through its first order derivatives (Eqs. (8) and (9)), being important only differences on the Φ values but not the Φ level. The Φ field is usually evaluated once its corresponding ϕ field is known (Table 2), the Φ values over the boundaries being obtained by integrating the flux of ϕ through the boundaries. An exception can be the evaluation of the flow field primarily by solving the streamfunction Eq. (17). Extending the procedure to all the domain boundaries, starting from any suitable reference point, leads to first kind boundary conditions for Φ over all the

Table 2
Coupling of ϕ and Φ and the diffusion coefficients for Φ for some usual situations

Physical principle	ϕ	Φ	$\Gamma_{\Phi,xx}$	$\Gamma_{\Phi,yy}$	$\Gamma_{\Phi,xy}$	Φ Contour plots
Overall mass conservation	1	ψ—Streafunction	$\frac{\mu K_{xx}}{\rho K^2}$	$\frac{\mu K_{yy}}{\rho K^2}$	$\frac{\mu K_{xy}}{\rho K^2}$	Streamlines
i species mass conservation	C_i	M_i—i species massfunction	$\frac{D_{i,xx}}{\rho D_i^2}$	$\frac{D_{i,yy}}{\rho D_i^2}$	$\frac{D_{i,xy}}{\rho D_i^2}$	i species masslines
Energy conservation	T	H— Heatfunction	$\frac{c_p k_{xx}}{k^2}$	$\frac{c_p k_{yy}}{k^2}$	$\frac{c_p k_{xy}}{k^2}$	Heatlines

$K^2 = K_{xx}K_{yy} - K_{xy}^2$, $D_i^2 = D_{i,xx}D_{i,yy} - D_{i,xy}^2$ and $k^2 = k_{xx}k_{yy} - k_{xy}^2$.

domain boundary. Starting from a reference point at the boundary, where $\Phi = \Phi_{\text{ref,b}}$, the value of Φ at a generic point P, b located at the boundary is evaluated as

$$\Phi_{\text{P,b}} = \Phi_{\text{ref,b}} + \int_{\text{ref,b}}^{\text{P,b}} \mathbf{J}_\phi \cdot \mathbf{n} \mathrm{d}s_{\text{b}} \tag{19}$$

When the boundary under analysis is not crossed by the flux $\mathbf{J}_\phi$ it is obvious that it is $\Phi =$ constant over such a boundary.

Due to their physical nature, ϕ and Φ are C^0 continua. Thus, at each point of any interface between contiguous portions 1 and 2 of the domain, even with different properties (as it is the case of conjugate heat and/or mass transfer problems) it is

$$\phi_1 = \phi_2 \quad \Phi_1 = \Phi_2 \tag{20}$$

Also due to its physical nature, $\Phi_1 = \Phi_2$ (or $\psi_1 = \psi_2$) guarantees the conservation of ϕ through such an interface.

Using the correct diffusion parameters, the field of a particular ϕ or Φ can be evaluated simultaneously in the overall domain, even if it is composed by portions with different transport properties [26].

3. Numerical modeling

From a numerical viewpoint, Eq. (6) is a convection–diffusion equation for anisotropic media. Due to the fact that, in general, $\Gamma_{\phi,xy} \neq 0$, there are the diffusive terms $\partial/\partial x(\Gamma_{\phi,xy}\partial\phi/\partial y)$ and $\partial/\partial y(\Gamma_{\phi,xy}\partial\phi/\partial x)$ involving ϕ explicitly that are not present when dealing with isotropic media. Thus, a most general numerical procedure was developed in order to directly account (in a fully implicit manner) for these additional diffusive terms, instead of its inclusion in a fictious source term for the ϕ equation.

The differential equation for Φ (Eq. (14)) (or for ψ) is a conduction type equation, also with the additional diffusive terms $\partial/\partial x(\Gamma_{\Phi,xy}\partial\Phi/\partial y)$ and $\partial/\partial y(\Gamma_{\Phi,xy}\partial\Phi/\partial x)$, which are absent when dealing with isotropic media. As the numerical procedures used for Φ are the same as used for ϕ, such terms are directly accounted for in a fully implicit manner and not taken as part of the source term of Eq. (14). It should be stressed that the Φ equation includes a true source term if the fluid flow subsists. The ψ equation (17) is also a conduction-type equation, treated in the same way as explained for Φ, and its source term vanishes only in the case of constant properties or absent gravitational effects.

The differential equations were taken in the conservative form, as they are to be solved through control-volume based numerical methods, designed for conserved variables. However, it should be retained that each particular Φ (or ψ) is not a conserved variable. This is not a serious problem, as its contour plots are used mainly for visualization purposes.

Variables p and ψ are not conserved variables, and it is from their fields that can be evaluated the velocity field. This velocity field does not strictly verify the mass conservation equation at the control volume level. The differential pressure diffusion equation (the last row of Table 1) is a reasonable description of the mass conservation equation, but it is well known that a Poisson equation for the pressure can well describe the mass conservation at a differential level but not at the control volume level [26,27].

When dealing with porous media, the pressure–velocity link is given by the Darcy flow model (Eq. (7)). The flow field can be evaluated from the pressure field, this one being known from the solution of the pressure diffusion equation (last row of Table 1), with the appropriate pressure boundary conditions. These are straightforward in closed domains or in open domains with known pressure or pressure gradient at the boundaries. The flow field can also be evaluated once the streamfunction field is known, which is obtained as the solution of the streamfunction diffusion equation (17), with the appropriate streamfuntion boundary conditions. In closed domains or in open domains with known velocities or mass fluxes at the boundaries the boundary conditions are easily specified. This is the way followed in the present work.

Once the adequate diffusion coefficients for ϕ, Φ and ψ are defined, even when the global domain includes portions with different transport properties, the conjugate heat and/or mass transfer problems are solved simultaneously in the overall domain for each particular variable, a procedure with noticeable advantages [26].

The numerical method used in this work is a two-dimensional laminar version of the control-volume based finite element method described in [28], adapted to include the additional diffusive terms present in anisotropic media in a fully implicit manner. To converge, the method is very sensitive to the relaxation factors, which need to be well adjusted by trial and error for each case. The diffusion coefficients are taken as constant over each finite element, implying the location of nodes at the interfaces when dealing with domains composed by portions of different properties. A uniform 101×101 mesh was selected after some preliminary testes of asymptotic type.

4. Illustrations

In order to illustrate the unified approach proposed, three types of heat transfer problems were considered: pure conduction, pure natural convection in a porous enclosure and conduction/convection conjugate problem. The mass transfer calculation of a particular chemical species is similar to the heat transfer calculation, and finding the masslines for such chemical species is similar to finding the heatlines. Thus, the proposed procedure is illustrated through heat transfer problems only. The overall domain under analysis, sketched in Fig. 1, is always a square of side length L, which can be uniform or composed by two (left and right) halves of different properties. The upper and lower boundaries are adiabatic and impermeable, and the heat transfer process occurs always from the isothermal hot left vertical boundary to the isothermal cold right vertical wall. The vertical boundaries are impermeable. The anisotropy angles are denoted as γ for thermal conductivity and as δ for permeability. As sketched in Fig. 1, such angles are measured between the horizontal direction x and the first principal direction ξ, which can be different for thermal conductivity (ξ, k) and for permeability (ξ, K).

The space co-ordinates are made dimensionless as $x_* = x/L$ and $y_* = y/L$, and the temperature is made dimensionless as $T_* = (T - T_C)/(T_H - T_C)$. Additional

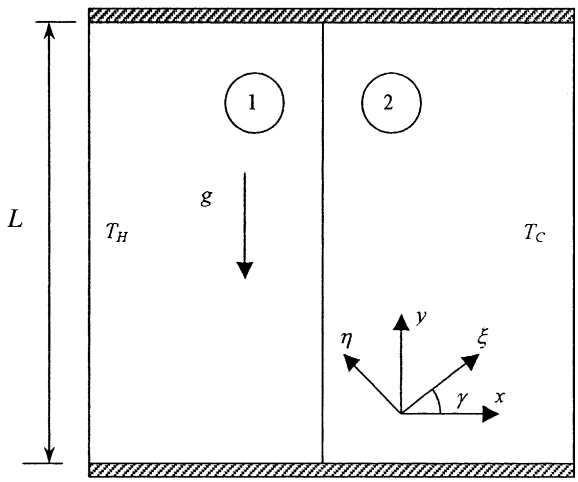

Fig. 1. Physical model and geometry.

dimensionless parameters governing the problem are the thermal conductivity anisotropy angles for media 1 and 2, γ_1 and γ_2, respectively, the ratio between the principal thermal conductivities $R_k = k_{\eta,k}/k_{\xi,k}$ for media 1 and 2, $R_{k,1}$ and $R_{k,2}$, respectively, and the ratio between the principal thermal conductivities of media 2 and 1, $R_{k,21} = k_{(\xi,k),2}/k_{(\xi,k),1}$. The heatfunction is made dimensionless as $H_* = H/[k_{(\xi,k),1}(T_H - T_C)]$, the reference value used to make H dimensionless being $k_{(\xi,k),1}(T_H - T_C) = k_{(\xi,k),1} \times L \times [(T_H - T_C)/L]$, the one-dimensional conduction heat flow that crosses all the domain in the x direction (by unit depth) under the thermal conductivity $k_{(\xi,k),1}$. The value of the heatfunction is set null over the lower adiabatic boundary of the domain. As the upper and lower boundaries of the domain are adiabatic, the heatfunction value gives very useful information about the global heat transfer crossing the domain [18]. Additional dimensionless parameters and reference values will be introduced when needed.

4.1. Conduction heat transfer

The domain is always composed by two-halves of different thermal conductivities, and the illustration of the procedure for pure conduction problems is given through the isothermals (left side) and heatlines (right side) for each case considered.

In Fig. 2a and b are presented the results for anisotropic medium 1 and isotropic medium 2, with different anisotropy angles for medium 1 and different thermal conductivity ratios $R_{k,1}$ and $R_{k,21}$. In Fig. 2c and d are presented the results for anisotropic media 1 and 2, with different anisotropy angles (with the same sign in both media) and different thermal conductivity ratio $R_{k,21}$. In Fig. 2e and f are presented the results for anisotropic media 1 and 2, with different anisotropy angles (with opposite signs) and different thermal conductivity ratios $R_{k,2}$ and $R_{k,21}$.

From the presented results, one can see clearly the influence of anisotropy on the temperature distribution and on the paths followed by heat when crossing the domain. The results suggest that particular arrangements of anisotropic materials can be made in order to have composite materials with the desired thermal performance, in terms of overall heat transfer, temperature distribution and heat path. It can be observed a marked difference in the slopes of the heatlines at the interface between media 1 and 2. This is the result of a combination of different anisotropy angles of the adjacent media (Fig. 2a and b, where medium 2 is isotropic, and Fig. 2d, e and f), and different thermal conductivity ratios $R_{k,1}$, $R_{k,2}$ and $R_{k,21}$ (Fig. 2a, b, d and f). For the results in Fig. 2e, only the effect of different anisotropy angles is present at the interface. In Fig. 2c there is no difference in the slopes of the heatlines at the interface, as media 1 and 2 have the same heat transfer properties.

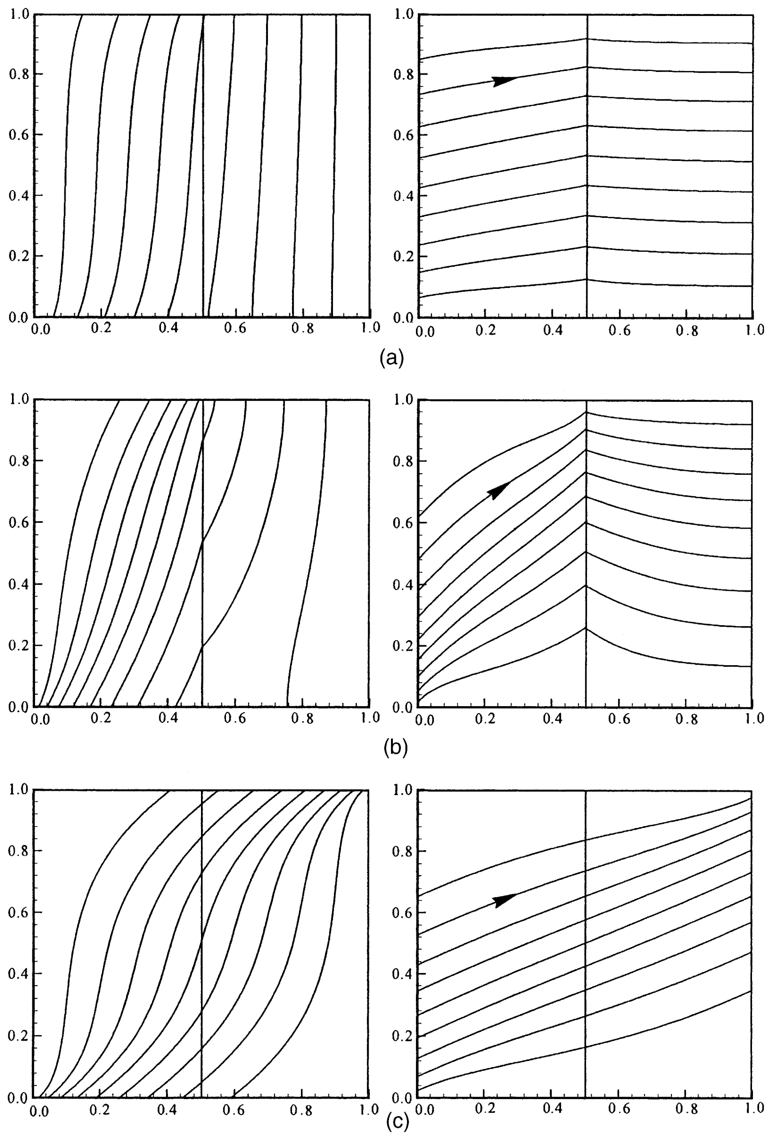

Fig. 2. Isothermals (left) and heatlines (right) for pure conduction problems: (a) $\gamma_1 = 30°$, $R_{k,1} = 0.5$, $R_{k,2} = 1.0$, $R_{k,21} = 1.0$, $\Delta H_* = 0.09$, $H_{*,\max} = 0.92$; (b) $\gamma_1 = 60°$, $R_{k,1} = 0.1$, $R_{k,2} = 1.0$, $R_{k,21} = 0.5$, $\Delta H_* = 0.03$, $H_{*,\max} = 0.29$; (c) $\gamma_1 = 30°$, $R_{k,1} = 0.1$, $\gamma_2 = 30°$, $R_{k,2} = 0.1$, $R_{k,21} = 1.0$, $\Delta H_* = 0.06$, $H_{*,\max} = 0.58$; (d) $\gamma_1 = 60°$, $R_{k,1} = 0.1$, $\gamma_2 = 30°$, $R_{k,2} = 0.1$, $R_{k,21} = 0.5$, $\Delta H_* = 0.02$, $H_{*,\max} = 0.22$; (e) $\gamma_1 = 30°$, $R_{k,1} = 0.1$, $\gamma_2 = -30°$, $R_{k,2} = 0.1$, $R_{k,21} = 1.0$, $\Delta H_* = 0.07$, $H_{*,\max} = 0.68$; and (f) $\gamma_1 = 60°$, $R_{k,1} = 0.1$, $\gamma_2 = -30°$, $R_{k,2} = 0.5$, $R_{k,21} = 0.1$, $\Delta H_* = 0.01$, $H_{*,\max} = 0.12$. $\Delta T_* = 0.1$ in all cases.

In summary, from the heatlines can be obtained a complete picture of the conduction heat transfer process taking place in anisotropic media and, in particular, at the interface between adjacent media of different thermal properties.

Conduction heat transfer in isotropic solids takes place in a way such that the heat flow vector has an opposite sense to the temperature gradient, but both vectors are perfectly aligned. In anisotropic media such vectors are not aligned, and they form an angle between 90° and 180°. This can be observed by superposing isotherms and heatlines for any case of Fig. 2a–f. At a selected point, the heat flow vector is tangent to the corresponding heatline, and the temperature gradient is normal to the corresponding isotherm and points in the direction of increasing temperature values.

It is also observed the perpendicularity of the isotherms relative to the adiabatic walls in the case of

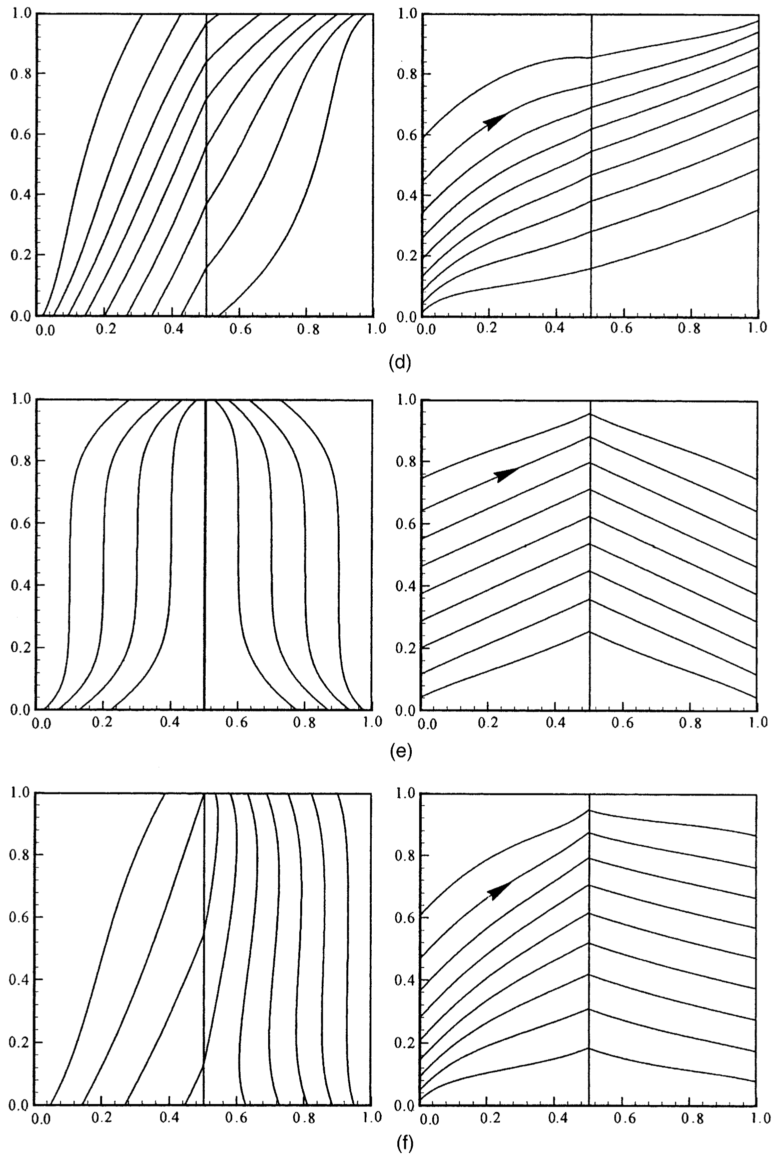

Fig. 2 (*continued*)

isotropic media. When the media are anisotropic, such perpendicularity does not exist. For isotropic media the heatlines are perpendicular to the isothermals, a behaviour that is not observed for anisotropic media.

4.2. Natural convection in a square porous enclosure

Additionally to the dimensionless variables and parameters introduced in the previous situation, the velocity components are made dimensionless as $u_* = u/(\alpha_{(\xi,k),1}/L)$ and $v_* = v/(\alpha_{(\xi,k),1}/L)$, and the streamfunction is made dimensionless as $\psi_* = \psi/(\rho\alpha_{(\xi,k),1})$. The streamfunction is null over all the impermeable boundaries of the domain. The permeability anisotropy angles for media 1 and 2, δ_1 and δ_2, respectively, the ratio between the principal permeabilities $R_K = K_{\eta,K}/K_{\xi,K}$ for media 1 and 2, $R_{K,1}$ and $R_{K,2}$, respectively, and the ratio between the principal permeabilities of media 2 and 1, $R_{K,21} = K_{(\xi,K),2}/K_{(\xi,K),1}$ are the dimensional parameters introduced by fluid flow in anisotropic porous media. The natural convection problem in the square porous enclosure under analysis introduces another dimensionless parameter, the Darcy Rayleigh number, defined as $Ra = (g\beta\Delta TK_{(\xi,K),1}L)/(\nu\alpha_{(\xi,k),1})$.

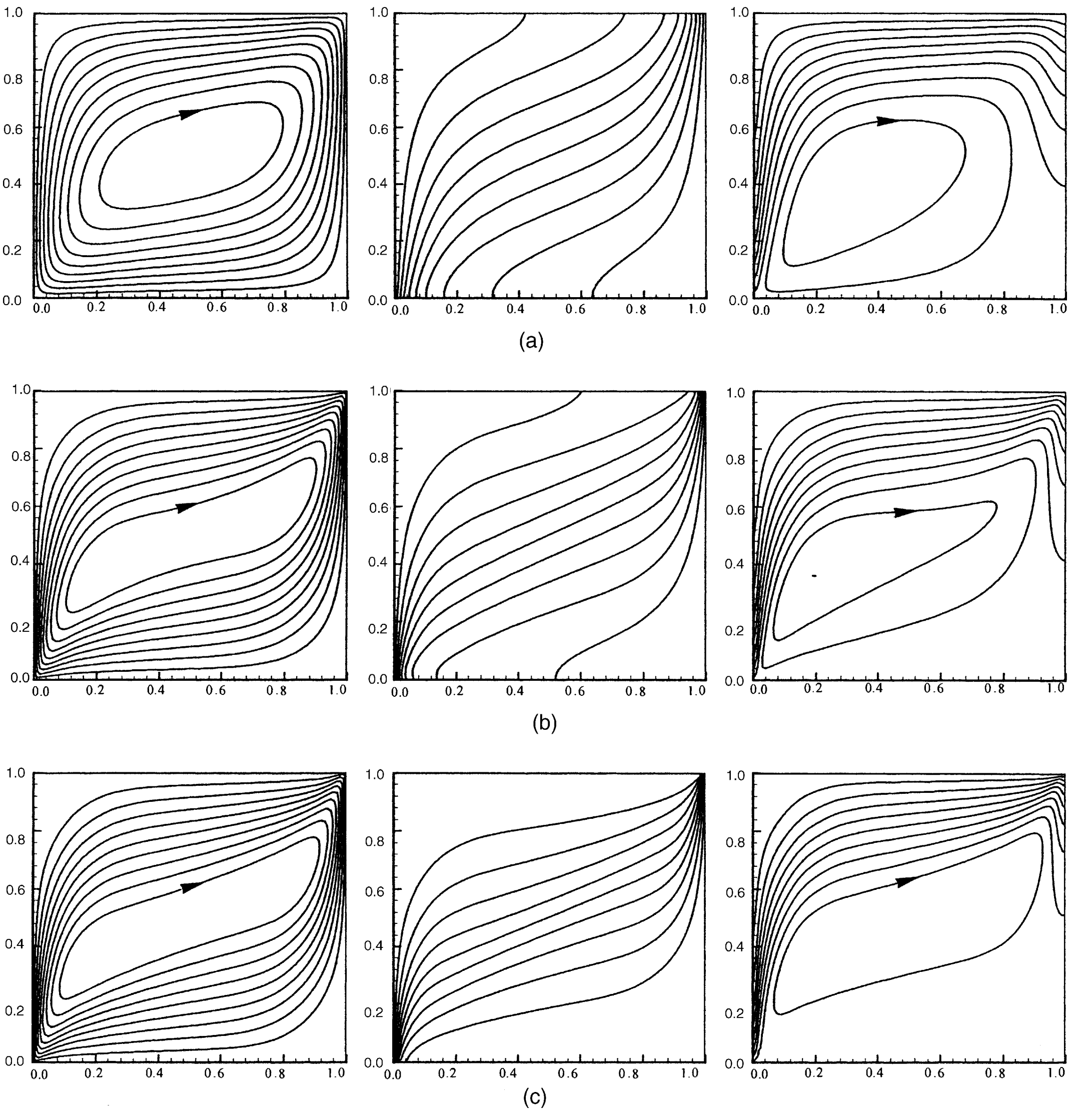

Fig. 3. Streamlines (left), isothermals (center) and heatlines (right) for natural convection in a square porous enclosure for $Ra = 100$ (a) $R_{k,1} = R_{k,2} = 1$, $R_{K,1} = R_{K,2} = 1$, $\Delta\psi_* = 0.47$, $\psi_{*,\min} = -4.72$, $\Delta H_* = 0.40$, $H_{*,\min} = -0.94$, $H_{*,\max} = 3.08$; (b) $R_{k,1} = R_{k,2} = 1$, $R_{K,1} = R_{K,2} = 0.1$, $\delta_1 = \delta_2 = 60°$, $\Delta\psi_* = 0.95$, $\psi_{*,\min} = -9.50$, $\Delta H_* = 0.81$, $H_{*,\min} = -2.15$, $H_{*,\max} = 5.95$; (c) $R_{k,1} = R_{k,2} = 0.1$, $\gamma_1 = \gamma_2 = 30°$, $R_{K,1} = R_{K,2} = 0.1$, $\delta_1 = \delta_2 = 60°$, $\Delta\psi_* = 0.92$, $\psi_{*,\min} = -9.17$, $\Delta H_* = 0.76$, $H_{*,\min} = -1.29$, $H_{*,\max} = 6.28$. $\Delta T_* = 0.1$ in all cases.

The two portions of the domain are made of the same material, that is, the properties are the same over all the domain. This corresponds to unit values of parameters $R_{k,21} = k_{(\xi,k),2}/k_{(\xi,k),1}$ and $R_{K,21} = K_{(\xi,K),2}/K_{(\xi,K),1}$, as well as to equal values for the parameters R_k and R_K, being $R_{k,1} = R_{k,2}$ and $R_{K,1} = R_{K,2}$. All the presented results correspond to the same value of the Darcy Rayleigh number $Ra = 100$.

In Fig. 3a are presented the results for isotropic media 1 and 2 both in what concerns thermal conductivity and permeability. This corresponds to the usual natural convection heat transfer problem in a differentially heated square porous enclosure. In Fig. 3b are presented the results for isotropic media in what concerns thermal conductivity and anisotropic in what concerns permeability. The anisotropy on permeability promotes the flow in directions with an angle of 60° relative to the horizontal. The hot fluid is more forced toward the neighboring of the upper-right corner, the 'hot' isotherms are more displaced to the right and the 'cold' isotherms suffer a slight displacement to the left. The thermal gradient is high near the upper-right

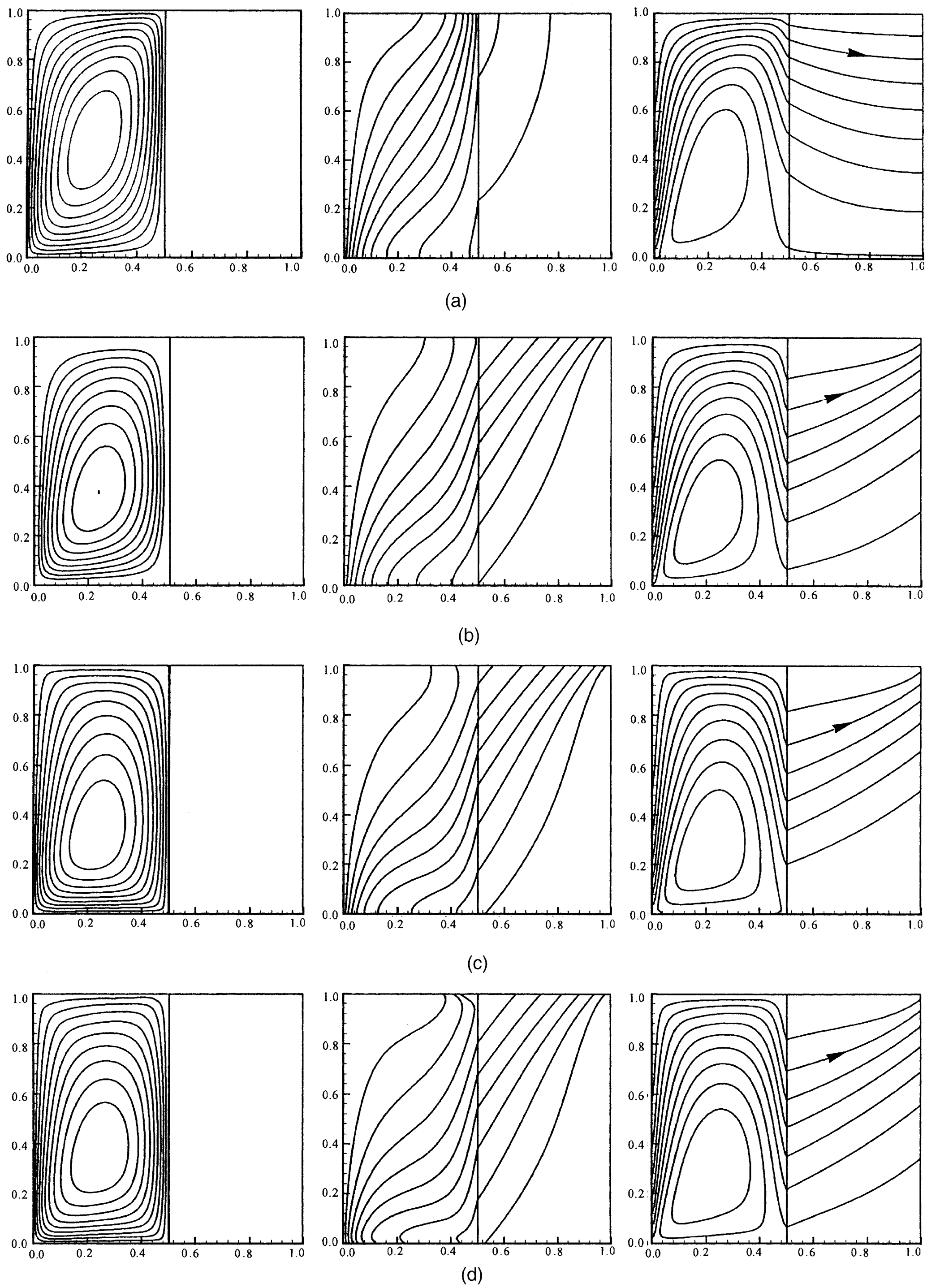

Fig. 4. Streamlines (left), isothermals (center) and heatlines (right) for natural convection in a rectangular porous enclosure (left-half of the domain) and pure conduction on the right-half of the domain, for $Ra = 100$ and $R_{k,21} = 10$: (a) $R_{k,1} = R_{k,2} = 1$, $R_{K,1} = 1$, $\Delta\psi_* = 0.40$, $\psi_{*,\min} = -4.04$, $\Delta H_* = 0.39$, $H_{*,\min} = -0.74$, $H_{*,\max} = 3.12$; (b) $R_{k,1} = 1$, $R_{k,2} = 0.1$, $\gamma_2 = 45°$, $R_{K,1} = 1$, $\Delta\psi_* = 0.31$, $\psi_{*,\min} = -3.12$, $\Delta H_* = 0.31$, $H_{*,\min} = -0.85$, $H_{*,\max} = 2.26$; (c) $R_{k,1} = 1$, $R_{k,2} = 0.1$, $\gamma_2 = 45°$, $R_{K,1} = 0.1$, $\delta_1 = -30°$, $\Delta\psi_* = 0.38$, $\psi_{*,\min} = -3.80$, $\Delta H_* = 0.37$, $H_{*,\min} = -1.10$, $H_{*,\max} = 2.56$; (d) $R_{k,1} = 0.1$, $\gamma_1 = -30°$, $R_{k,2} = 0.1$, $\gamma_2 = 45°$, $R_{K,1} = 0.1$, $\delta_1 = -30°$, $\Delta\psi_* = 0.35$, $\psi_{*,\min} = -3.50$, $\Delta H_* = 0.33$, $H_{*,\min} = -0.86$, $H_{*,\max} = 2.47$. $\Delta T_* = 0.1$ in all cases.

corner, with more intense heat transfer in that region. In Fig. 3c are presented the results for anisotropic media in what concerns both thermal conductivity and permeability. In this case, remains valid the explanation made for Fig. 3b for the anisotropy on permeability. Thermal conductivity anisotropy, with an angle of 30°, promotes conduction heat transfer to the right side of the enclosure. The 'hot' isotherms are forced towards the cold wall and the 'cold' isotherms are forced towards the hot wall, with intense thermal gradients near the upper-right and lower-left corners of the enclosure.

It is clear from the Fig. 3a–c the importance of anisotropy over the transport phenomena taking place, as well as the valuable pictures given by the streamlines and by the heatlines for a most complete understanding and analysis of the problem. From Fig. 3c it is observed how the combined anisotropy of both the thermal conductivity and permeability can concentrate the heat transfer path followed by heat transfer near the upper-right corner of the enclosure. The difference of the dimensionless heatfunction between the upper and lower horizontal adiabatic walls is the global Nusselt number for the enclosure, evaluated as $\dot{Q}'_{\text{convection}}/[k_{(\xi,k),1} \times (T_{\text{H}} - T_{\text{C}})]$.

4.3. Conjugate conduction/convection heat transfer

In this case, the left-half of the domain is a rectangular enclosure with an aspect ratio 1/2, and the right-half of the domain is a solid. The dimensionless variables and parameters introduced in the previous situations are sufficient to describe the present situation. All the presented results correspond to the same values $Ra = 100$ and $R_{k,21} = 10$.

In Fig. 4a are presented the results for isotropic medium 1, both in what concerns thermal conductivity and permeability, and isotropic solid medium 2. In Fig. 4b are presented the results for isotropic medium 1, both in what concerns thermal conductivity and permeability, and anisotropic solid medium 2. In Fig. 4c are presented the results for isotropic medium 1 in what concerns thermal conductivity and anisotropic in what concerns permeability, and anisotropic solid medium 2. In Fig. 4d are presented the results for anisotropic medium 1 in what concerns both thermal conductivity and permeability, and anisotropic solid medium 2.

Analysis of Fig. 4a–d shows the importance of anisotropy over the transport phenomena taking place, which cannot be neglected in many practical situations. The streamlines and the heatlines show to be very effective tools to visualize the heat and mass transfers occurring in the considered domain, and the proposed unifying procedure shows to be very effective to deal with conjugate conduction/convection heat transfer problems.

5. Conclusions

The main conclusion is that the streamfunction, heatfunction and massfunction for two-dimensional domains of (heat and mass transfer) anisotropic media can be unified both from the physical and numerical viewpoints. Mass transfer in saturated porous media has been considered through the Darcy flow model. Evaluation of a primitive conserved variable ϕ is a problem formally similar to the evaluation of its related function Φ, whose contour plots are used for visualization purposes.

Attention needs to be given to the diffusion coefficients of the streamfunction, heatfunction and massfunction, which are uniquely obtained from the diffusion coefficients of the corresponding primitive variables. This is important in media of constant properties, and crucial in conjugate heat and/or mass transfer problems, occurring in domains with portions of different (or even very different) transport properties.

Transport phenomena in anisotropic media include additional diffusive terms in the differential equations, which are not present when dealing with isotropic media. The same is true also for the streamfunction, heatfunction, and massfunction when dealing with anisotropic media. All the diffusive terms can be considered in a fully implicit way, the numerical solution being obtained simultaneously in the overall domain even if it composed by portions with different transport properties. The numerical procedures and code routines designed for isotropic media need only slight modifications to deal with anisotropic media through fully implicit consideration of the aforementioned additional diffusive terms.

The proposed unification is very useful from the physical and numerical viewpoints. The primitive conserved variables and the streamfunction, heatfunction and massfunction fields can be evaluated by using the same procedures and code routines, primarily designed to evaluated the fields of primitive conserved variables. In the case of fluid flow in saturated porous media, the streamfunction is useful for visualization purposes and it is also frequently the field from which the velocity field is evaluated.

Contour plots of streamfunction, heatfunction and massfunction are the effective tools to visualize the paths followed by the involved transport phenomena, as well as to quantify such transport phenomena. When expressed in dimensionless form, the streamfunction, heatfunction and massfunction are closely related with the dimensionless transfer parameters such as the Nusselt and/or Sherwood numbers. Illustration results are very clear about the capabilities and usefulness of the proposed unified approach for problems ranging from pure conduction heat transfer to conjugate conduction-convection heat transfer. This approach was designed to deal with anisotropic media, the case of isotropic media emerging as a particular case.

As many of the actually used materials are anisotropic, this unification is of special importance for practical purposes, both to evaluate the transport phenomena occurring in anisotropic media as well as to visualize such transport phenomena using the most effective tools for that.

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